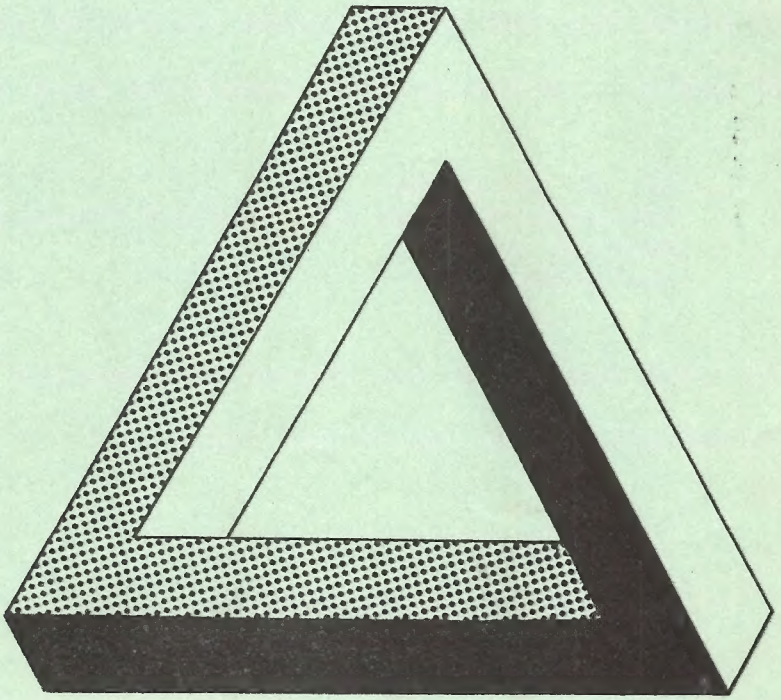
TRIGON



SCHOOL MATHEMATICS JOURNAL
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EDITORIAL COMMITTEE

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All material which is suitable for publication will be acknowledged in the Journal.

CARL FRIEDRICH RUINS THE SCHOOLMASTER'S NAP

This is a true story about the young Carl Friedrich Gauss, who became one of the greatest mathematicians in history. It was written as the introduction to a booklet on arithmetic progressions which was never published, unfortunately. The author, Professor Rosenbaum, is one of those rare individuals who is both a mathematician and truly great teacher. He was chairman of the mathematics department at Wesleyan University in Middletown, Connecticut, but has forsaken the classroom to become the university vice-president for academic affairs and acting president. For several years Professor Rosenbaum was editor of the American Mathematical Monthly, the official publication of the Mathematical Association.

In the big schoolroom in Braunschweig, Germany, everyone was sleepy on this warm sunny afternoon. It was early summer of 1785, and the children were looking forward eagerly to vacation. During the lunch hour they had eaten as fast as they could, and then had run out to play games. By the time the school bell had rung for the start of the afternoon session, they were pretty tired; and, after they had settled down at their seats, they soon became drowsy.

But the sleepiest of all was Herr Buttner, the schoolteacher. In this one-room school, he had to preside over about one hundred pupils, ranging in age from seven to fourteen, and they were more than a handful. A good many of the pupils were merely marking time until their confirmation date, when they would happily quit school and go to work. Herr Buttner had one powerful aid in keeping order and instilling learning - a big switch, which he used freely on the backs of his pupils to punish mischief or ignorance. The teacher had had a busy morning: hearing lessons in turn, walking up and down the aisles to ensure diligent attention to work, and exercising his strong right arm from time to time to restore wavering interests in study. Herr Buttner now regretted the heavy lunch he had eaten; as the afternoon

session began, he sat behind his desk with his head in his hands, wishing with all his heart that he might take a little nap.

Unfortunately, he was faced with one of his hardest jobs: teaching arithmetic. Each child began the study of arithmetic at the age of nine, and continued with it until Herr Buttner thought he had mastered the subject, a state of affairs which almost never was reached before the pupil attained the age of fourteen and escaped from Herr Butner's clutches. Thus, most of the students were in the arithmetic class. Moreover, they were at all stages of advancement. If only there were some way to give them all something to do, without the teacher's having to expend much energy! Herr Buttner rubbed his hands over his face - and had an inspiration.

"Ah", he thought to himself. "This'll keep 'em busy for a while."

Then he cleared his throat. "Boys!" he bellowed. "Get out your slates." The children groaned. They didn't enjoy Herr Buttner's problems under the best of circumstances, and, on this warm afternoon, they would much have preferred hearing a story, or reading a book, as the pre-arithmetic group was to do - anything would have seemed better than the arithmetic problem which they knew was coming.

"All right, boys: here's what you do," ordered Herr Buttner. "Add up all the whole numbers from one to two hundred."

There was an immediate reaction from the class, subdued out of respect for the schoolmaster's switch. Under their voices the pupils whispered angrily: "All the numbers from one to two hundred?" "That's unfair!" "What a job!" "That'll take all afternoon!"

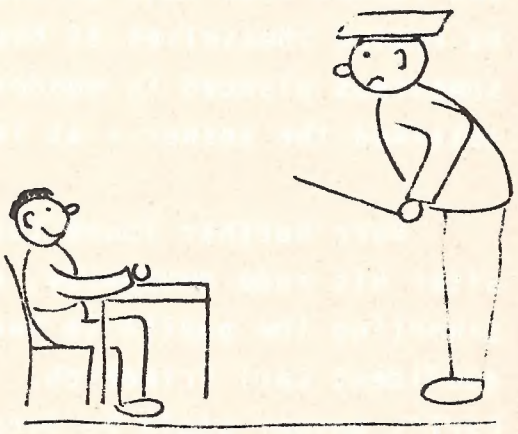
Herr Buttner smiled with satisfaction. He foresaw a

long period of relaxation for himself. "Quiet!" he roared. "You heard what I said: one plus two plus three plus and so forth, up to two hundred." He continued in a sharp voice, "Now get busy; don't waste any time."

The teacher glanced about the room to check that every head was bent over a slate. Then he leaned back in his arm chair, yawned, settled himself comfortably, and closed his eyes. He took two or three deep, slow breaths, when ...

"Clack!" he was jarred awake by a sharp report. He brought his eyes into focus, and saw a slate upside down, resting on his desk. He turned his eyes to see what fool had done this, and there was Carl Friedrich Gauss, a quiet boy of nine standing by his desk.

Carl Friedrich inclined his head toward his slate. "There it is," he stated simply. The custom was to make a contest of every problem. The first child to get an answer placed his slate, upside down, on the schoolmaster's desk. The next to finish placed his slate, also upside down on top of the first one, and so on. When all the slates were piled up, the whole stack was turned over and the owner of the first to have the right answer was declared the winner. Carl Friedrich seemed to imply that he had already obtained the answer to this lengthy problem in addition - in just a couple of moments!



Herr Buttner glared directly at the quiet boy. He was interfering with the teacher's rest, a serious matter. If it had been anyone else in the school, Herr Buttner would have been sure that he was either a complete fool, or was being a "wise guy". But Carl Friedrich was neither a fool nor a jokester. In the small town of Braunschweig,

stories of unusual events soon became common knowledge, and the teacher had heard of what had happened when Carl Friedrich was four years old: the little boy had been watching his father, a mason contractor, figuring up the weekly wages of his various workmen. Tugging at his father's sleeve, Carl Friedrich had pointed out to him where he had made a mistake in his computation. So there was a slight chance that the boy had devised some shortcut for the problem of adding up the whole numbers from one to two hundred.

On the other hand, Carl Friedrich had appeared to be merely a docile, hard-working pupil in his two years in school, exhibiting no signs of extraordinary talent; and now, at the age of nine, he was just beginning arithmetic - he hadn't had time to learn much from Herr Buttner. In any case, the teacher didn't want to be cheated out of his rest.

"Sit down, boy", he growled. "We'll look at your slate later."

So, Carl Friedrich returned to his bench, folded his hands and sat, quiet and thoughtful, while the others struggled with the addition. Occasionally, he smiled to himself as his neighbours muttered when they made mistakes and rubbed them out, as they whispered under their breaths to remind themselves of how much to "carry", and as they sometimes glanced in wonderment at the little boy who had obtained the answer - at least an answer - so quickly.

Herr Buttner found that he couldn't enjoy his rest, after his rude awakening, so he went up and down the aisles, exhorting the pupils to hurry, and glaring at the self-confident Carl Friedrich. He probably half-wished that Carl's answer would be incorrect; then he might employ the switch to discipline the boy's impertinence.

Finally, after a long time had passed, another slate was placed on Carl Friedrich's; and, in another ten minutes,

half a dozen more were piled on. When most of the slates were in, Herr Buttner went to the desk and turned the pile over. All eyes were on his as he picked up Carl Friedrich's slate, on which a single number was written: 20,100. The teacher recognized this as the correct answer, and gave a nod of acknowledgement to the boy. The other slates were quickly checked: there were some correct answers, but most of the pupils had lost their footings in the morass of addition, and had come up with incorrect values for the sum.

According to report, Herr Buttner soon appreciated the remarkable qualities of Carl Friedrich Gauss, atoning for his early skepticism and sarcasm by sending to the big city of Hamburg for the best arithmetic book available, which the teacher provided from his own meager funds for the boy.

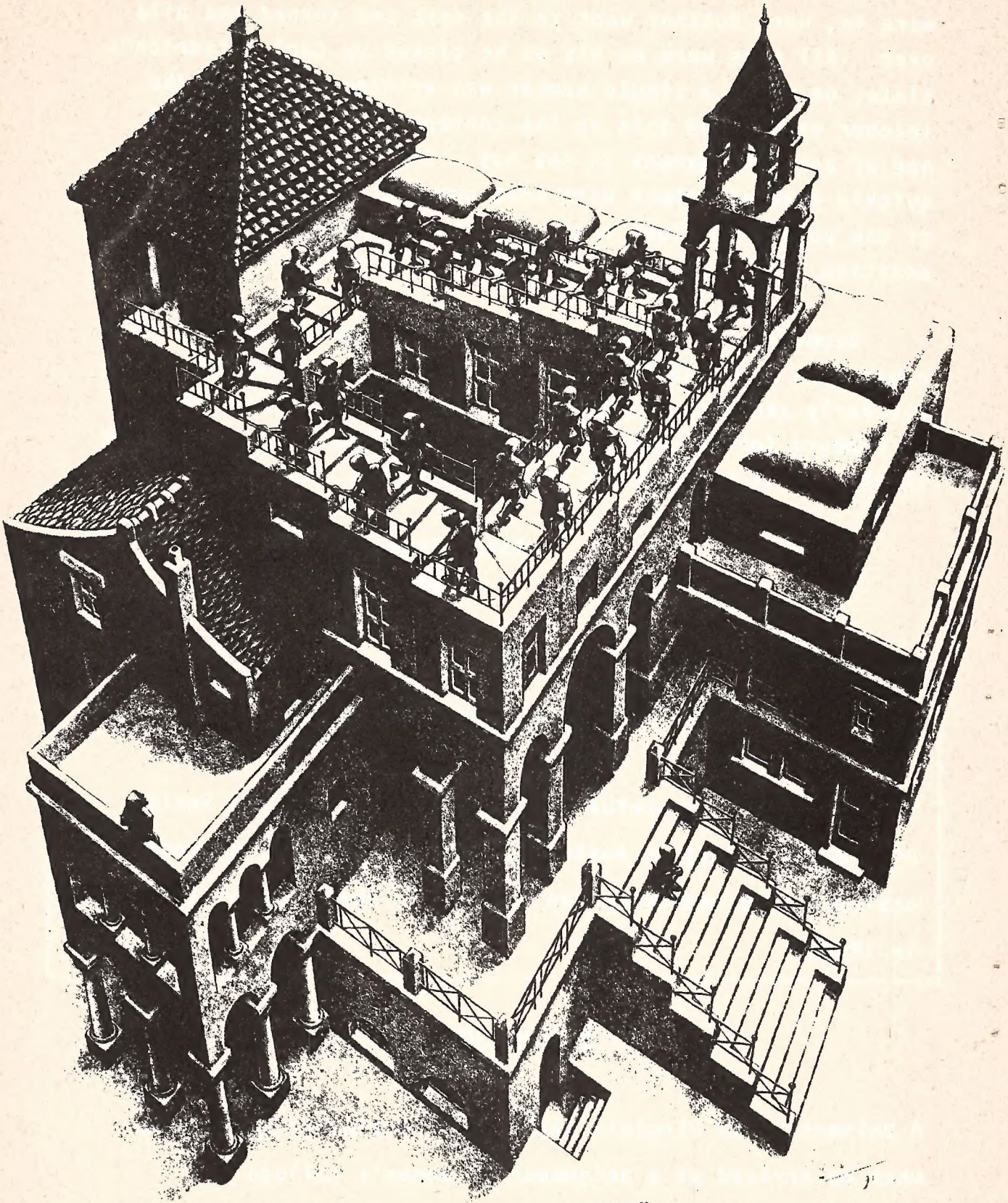
Taken from Odd Angles
Charles F. Linn
Doubleday

Acknowledgement

The Editors are grateful to the staff of Hollywood Senior High School, Western Australia who have (perhaps unknowingly!) assisted with the selection of some of the articles appearing in TRIGON.

College Daze

A University of Virginia mathematics professor was amazed when he arrived at a neighbouring women's college to lecture on "Convex Sets and Inequalities", to find the auditorium filled to capacity. A glance at the campus newspaper gave him the answer. His subject was announced as "Convicts, Sex and Inequalities".



"ASCENDING AND DESCENDING"

by M.C. Escher

Reprinted from - The M.C. Escher Calendar (Random House Inc., Publishers)

CHESS COLUMN

J. van der Hoek

For the World Chess Championship match this year, Bobby Fischer demanded certain conditions under which this match should be played. The International Chess Federation (FIDE) decided to limit the number of games to be played in the match to 36, and the winner of the match would be the first person to score 10 wins (draws not counted) or to have the best score after the 36th game. Fischer, however, demanded

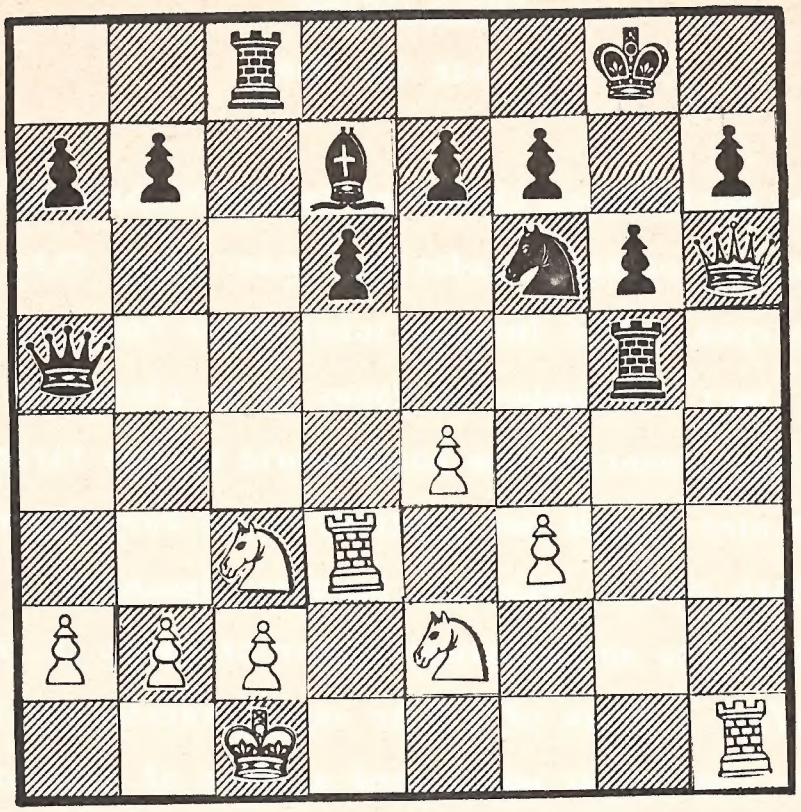
(a) that there be an unlimited number of games in the match and

(b) that after a score of 9-9 he should retain the title, meaning that the challenger would need a score of 10-8 or better to win the match. FIDE did not agree to these latter requirements of Fischer, so he relinquished his title. The new world Chess Champion is now Anatoly Karpov, a 23 year old University student from the Soviet Union. Here is one of his games against Victor Korchnoi, also of the Soviet Union, who ranks third in the world. It is from the 1974 candidates match.

WHITE: Karpov BLACK: Korchnoi

Sicilian Defence.

1.	P-K4 ,	P-QB4	11.	B-N3 ,	N-K4
2.	N-KB3,	P-Q3	12.	0-0-0 ,	N-QB5(c)
3.	P-Q4 ,	P×P	13.	B×N ,	R×B
4.	N×P ,	N-KB3	14.	P-KR5(d),	N×P
5.	N-QB3,	P-KN3(a)	15.	P-KN4 ,	N-B3
6.	B-K3 ,	B-N2	16.	N(Q4)-K2(e),	Q-R4
7.	P-KB3,	N-QB3	17.	B-R6(f),	B×B
8.	Q-Q2 ,	0-0	18.	Q×B ,	KR-QB1
9.	B-QB4,	B-Q2	19.	R-Q3 ,	R(B5)-QB4
10.	P-KR4(b),	R-QB1	20.	P-KN4!!,	R×P(g)



(diagram 1.)

21.	R-Q5! ,	R×R(h)
22.	N×R ,	R-K1
23.	N(K2)-B4(i),	B-B3(j)
24.	P-K5!!,	B×N(k)
25.	P×N,	P×P
26.	Q×RPch,	K-B1
27.	Q-R8ch	Resigns.(ℓ)

Notes. (a) The Dragon Variation (b) White intends to open the KR-file by eventually playing P-KR5 (c) This manoeuvre is important to rid white of his powerful white-squared bishop (d) A typical sacrifice to open the KR-file (e) Threatening 17. P-K5, P×P; 18. P-KN5, N moves; 19 Q×B (f) This forces the exchange of Black's powerful bishop (g) If 20....N-R4; 21. N-KB4, R×P (not N×N??); 22. N-Q5 and now white threatens both 23. Q×R and 23. N×Pch (h) not 21...N×R; 22 Q×Pch, K-B1; 23 Q-R8 mate (i) with the idea of 24. N×N ch, P×N; 25. N-Q5!; not 25. Q×Pch because the King will escape via K2 (j) To answer 24. N×N, P×N; 25. N-Q5 with 25....B×N (k) If 24....P×P; 25 N×Nch, P×N; 26 N-R5! and if 26....P×N; 27. R-N1ch etc. (ℓ) If 27....K-K2; 28. N×Bch, Q×N; 29. R-K1ch decides the matter.

In the next diagram white moves and mates black in three moves. This position looks very simple but do not be deceived.

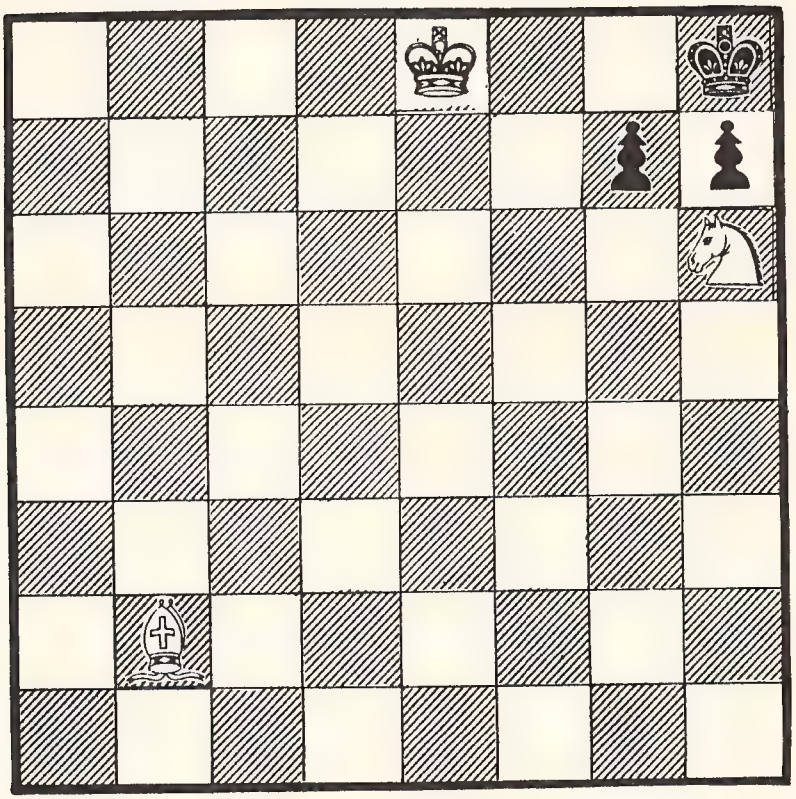


diagram 2.

Six little zeros became lost in a deep dark forest.

0 0 0 0 0 0

But a brave one found them and led them to safety. "My goodness," cried one zero. "Isn't he a fine one?" "Indeed he is," replied another zero. "He's one in a million."

THALES MEASURES A PYRAMID

You may have read, one place or another, about Euclid's encounter with an urchin on the steps of the library of Alexandria. The lad had all kinds of embarrassing questions to ask the mathematician about his book. But, it seems that Euclid was not the first Greek mathematician to run into this problem. Thales, who lived several centuries before Euclid, tangled with an uninhibited youngster on one of his trips to Egypt. Of course, all the kids had to do of a summer day was sit around and ask embarrassing questions of the visiting mathematician!

Anyway, it seems that friend Thales wandered down to Egypt one year to see the sights and, incidentally, try to learn a bit about the very practical work the Egyptians did in Geometry. He became particularly intrigued with the pyramids and the problem of measuring the height of a pyramid. The direct approach was "out", since even if he climbed to the tip of the pyramid there was no way of measuring straight down. And, anyway, the Greeks were not much inclined toward direct approaches when they could work out a nice mathematical solution. So, Thales sat down to give the matter some thought.

After a number of weeks spent thinking and calculating, he thought he had the problem solved, so he gathered up his drawing board, pencils, papyrus and a few instruments he had built, and went out to the pyramid.

Setting up shop there under his beach umbrella, Thales was working busily, when a small boy wandered up. "Whaddarya doin' with all that stuff out in the hot sun?" he asked.

"I'm calculating the height of yonder pyramid", replied Thales.

"Oh", said the boy. "Looks like the hard way".

Thales wasn't really listening to what the boy said, and continued his calculating, muttering to himself. "Hmmm.....hummm.....tangent.....confounded cheap pencils .....now, where's the table.....hummmmm.....multiply by 1.414.....drat.....someone should invent a ball-point pen .....bah.....made a mistake somewhere....."

About that time the boy interrupted this monologue. "I don't see why you need all these numbers and things".

Thales mumbled on. "Hmmmm, let's see.....no.....go away lad.....no, wait....I'll explain. You see all these numbers are related to the angles-this is a new thing-just invented it. Think I'll call it 'trigonometry'. You see, you get the angle on the top of the pyramid with this instrument here-just invented that too-and then you look up in the table.....and you measure the distance.....and..... drat it, you've got me all confused. Anyway, you measure this distance.....and then.....I think.....you multiply it by this number which is the cosine of the angle.....No,..... think I'll call this the 'tangent', and then you'll have the height of the pyramid,.....drat it.....something's wrong .....hmmmm.....confounded pencils....."

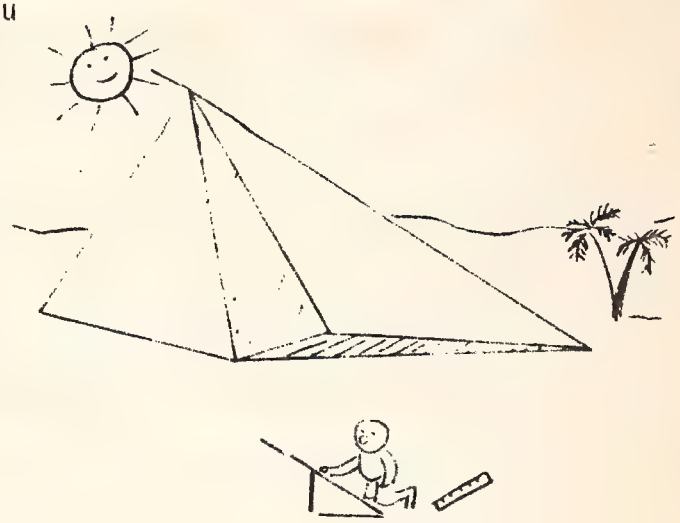
"I don't see it at all", the boy observed. "Still looks like the hard way to me".

"Look, kid", Thales broke in sharply. "This trigonometry is a complicated business. Why, I'm inventing a whole new branch of mathematics. This isn't for the likes of you. Now, go away and let me figure.....hmmm.....maybe sine of 34 degrees.....bah.....that couldn't be right....."

The lad was not so easily put off. "But look, Mr. Thales, why don't you just measure the shadow of the pyramid, and you'd be all set".

"What's that about the shadow?" Thales interrupted. "You've gotta have trigonometry for this. Now, go away".

"Okay, okay", said the boy. "Don't be so touchy. But if you measured the shadow of one of your long sticks there.....and you can measure the length of the stick.....you could figure out the height of the pyramid by a simple proportion-shadow of the pyramid is to the height of the pyramid as the shadow of the stick is to the height of the stick-simple, huh?"



"Hmph", Thales snorted, ".....a crude method.....lessee .....tangent of 34 degrees.....what did you say, boy? Couldn't be that easy.....now go away. I've work to do....."

So the lad went off to draw a few diagrams in the sand. And, Thales returned to his calculations and mutterings. Later in the day, however, he was seen out beside the pyramid with a stick in the ground, and his measuring rod on the ground.

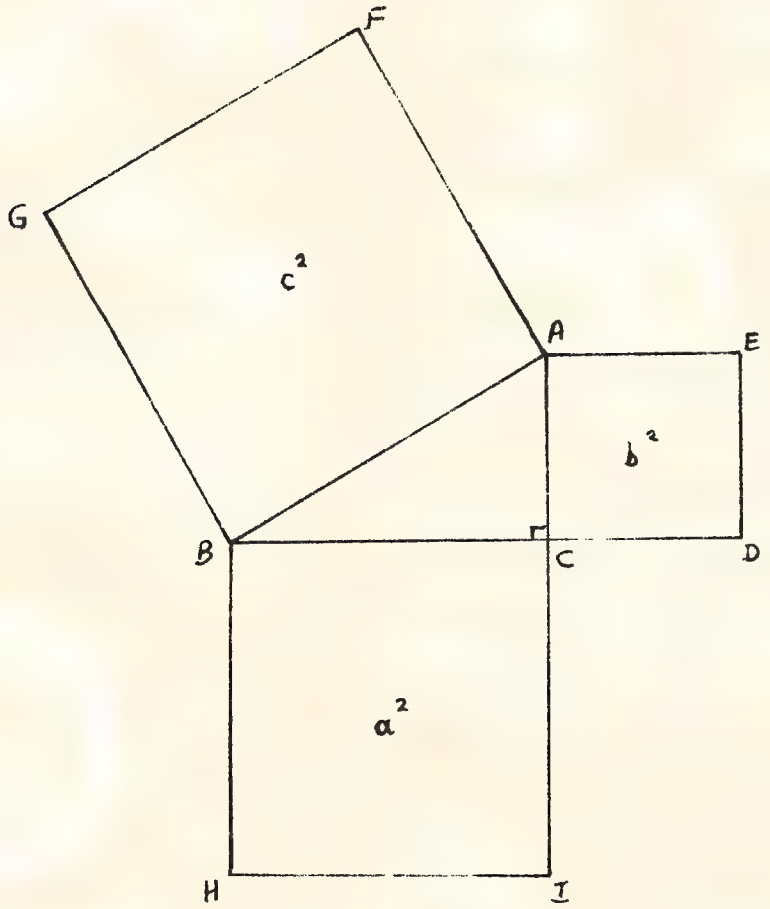
Evidently his results impressed the folks back home, for Thales feat of measuring the pyramid is much acclaimed in the history books.

And, trigonometry had to wait a few hundred years or so to be invented.

Adapted from Odd Angles
Charles F. Linn
Doubleday and Company.

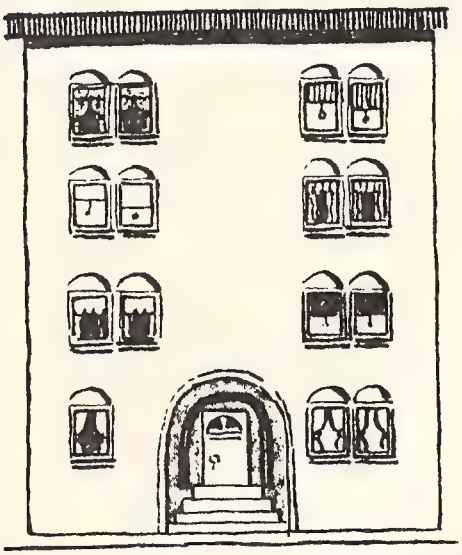
A DEMONSTRATION OF A PROOF OF THE PYTHAGOREAN THEOREM

By the Pythagorean Theorem, in Δ ABC where $m\angle ACB = 90^{o}$, $c^2 = a^2 + b^2$. But c^2 is the area of the square drawn on the hypotenuse and similar interpretations can be given to a^2 and b^2.



If we look at the square regions, the area of square region ABGF must equal the sum of the other two. If we appropriately cut the region ABGF into pieces, the pieces should fit over the square regions AEDC and BCIH. Try this demonstration with pencil and paper. (Hint: Draw the medians of the square.)

it's about time

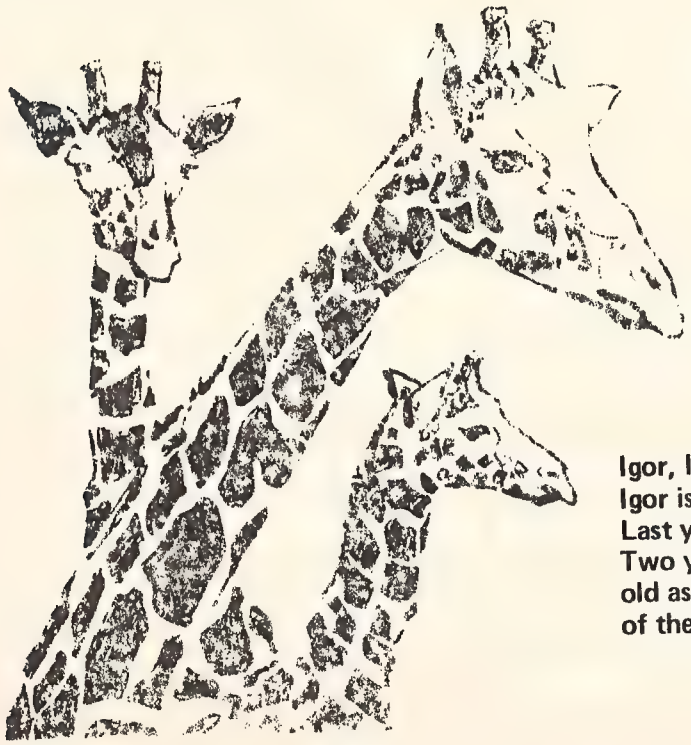


(1) Joe, Jack, John and Tom live one on each floor of a four-story apartment.
(2) Their ages are 10, 9, 8, 5, but not necessarily in that order.
(3) Joe lives directly above the 9-year-old and directly below the 8-year-old.
(4) Jack has to pass by the 5-year-old to leave the building from his apartment.
(5) Jack is more than one floor away from Tom, who is more than one year younger than Jack.

Find the ages and on which floor each of the boys lives.

If Alice and Betty can complete their jobs in 2 hours and if Alice and Charlene can do the same work in 3 hours, while Betty and Charlene require six hours to do the same jobs, how long would it take each girl working alone?

An hour ago, the time was four times as many minutes past 1:30 as there now are minutes until four o'clock. What time is it?



Igor, Ivan and Dornatz are all giraffes. Igor is as old as Ivan and Dornatz together. Last year Ivan was twice as old as Dornatz. Two years from now Igor will be twice as old as Dornatz. What are the respective ages of the three giraffes.

REPRINTED FROM MATHEMATICS STUDENT

A EUCLIDEAN FALLACY

Dr. Theodore H. MacDonald

Secondary students, and sometimes even their teachers, are prone to the idea that geometric proofs (because of the precise language employed) are somehow fool-proof. Of course, the fallacious proof that every triangle is isosceles is well known, but the common view is that the fallacy there is obvious.

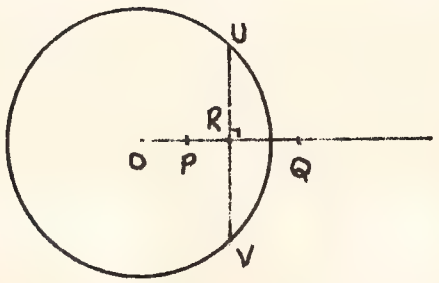
However, with the recent growth in an appreciation of the abstract, illogical statements made in a euclidean context seem prone to acceptance! For instance, consider the following:

Proposition: Every point in a circular region lies on its boundary.

Given: A circle of centre O, radius r, and an arbitrary point P inside the region bounded by the circle.

Construction: Let Q be the point on OP produced beyond P such that $OP \cdot OQ = r^2$.

Let the perpendicular bisector of PQ cut the circle circumference at U,V . Denote the middle point of segment PQ by R .



Dr. MacDonald is Chairman and Head of the Department of Mathematics, Melbourne State College.

Proof: $OP = OR - RP$

$OQ = OR + RQ$

$\therefore OQ = OR + RP$ because $RQ = RP$ (Construction)

$\therefore OP \cdot OQ = (OR - RP)(OR + RP)$

$= (OR)^2 - (RP)^2$

$= (OU^2 - RU^2) - (PU^2 - RU^2)$ (Pythagoras)

$= OU^2 - PU^2$

$= OP \cdot OQ - PU^2$ $(OP.OQ = r^2 = OU^2)$

$\therefore PU = 0$

$\therefore$ P is at U, on the circumference.

An Algebraic Analysis: The nature of the fallacy involved is most succinctly exposed by a refreshingly simple excursion into algebra.

Let $OP = p$

so that $OQ = \frac{r^2}{p}$

Now $(r - p)^2 > 0$ because the left term is a perfect square

$\therefore r^2 - 2rp + p^2 > 0$

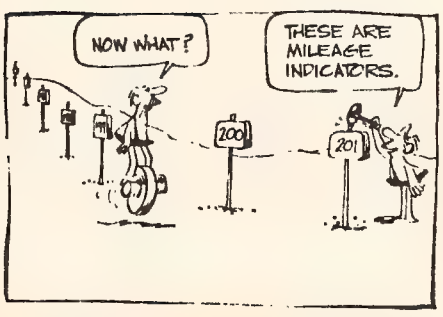
so $p^2 + r^2 > 2rp$.

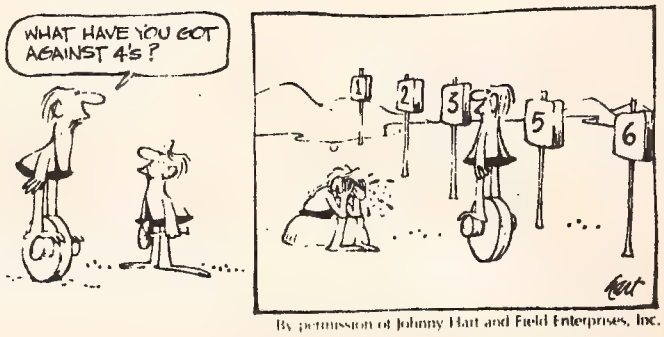
Divide through by $2p$, which is positive.

$$\tfrac{1}{2}(p + \frac{r^2}{p}) > r$$

$\therefore$ $OR > r$, since $OR = \tfrac{1}{2}(OP + OQ)$.

That means that R lies outside of the circle and so the points U and V do not exist and hence no calculations involving them can be made.





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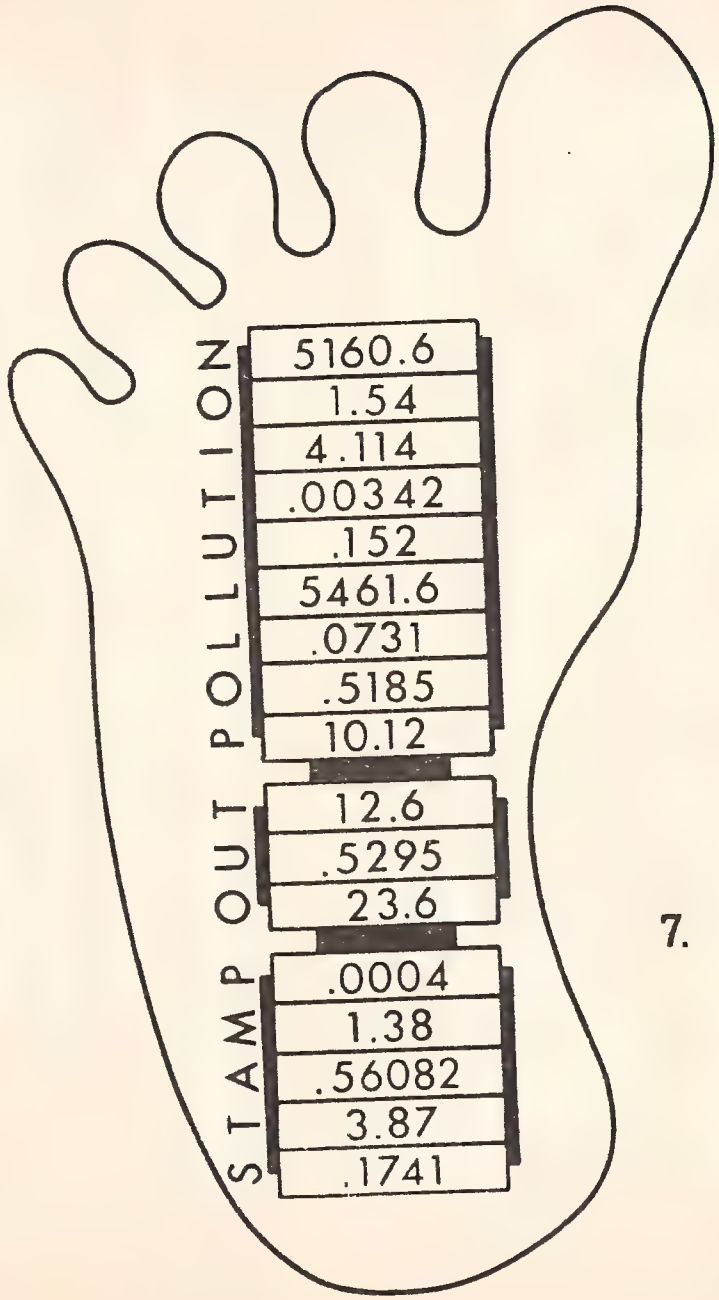

HIDDEN MUSIC

THERE IS HIDDEN MUSIC IN THE MESSAGE TO THE LEFT, TO FIND IT, FOLLOW THESE DIRECTIONS:

FIRST, work each problem and find the correct answer in the rectangles to the left.

SECOND, shade out the letter next to each correct answer. The HIDDEN MUSIC will remain!

1. $2.3 \times .6$

2. $3.08 \times .5$

3. $.019 \times 8$

4. 4.6×2.2

5. $90 \times .14$

6. $.86 \times 4.5$

7. $.17 \times .43$

8. $.005 \times .08$

9. 1.21×3.4

10. $3.05 \times .17$

11. $.036 \times .095$

12. $.719 \times .78$

13. 846×6.1

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A PIECE OF STRING

A piece of string dropped on a table has the shape shown in Figure 1. Travelling along the string starting at A there are three 'over crossings' followed by three 'under crossings'.

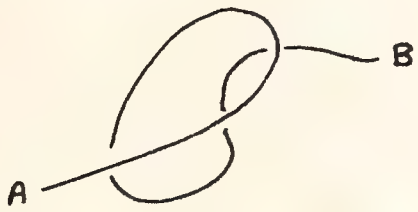


Fig. 1

This suggests that we can describe the string loop by the 'word' OOOUUU. What word describes the loop traversed in the opposite direction from B to A ?

Figure 2 shows some more string loops and their associated words.

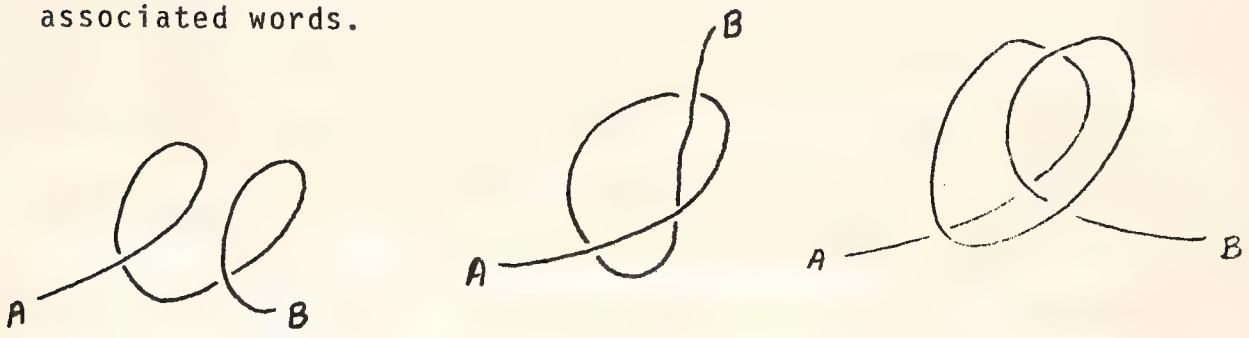


Fig. 2

Looking at the words generated so far we notice that:

(a) they have an even number of letters, and further, each word has equal numbers of Os and Us ;

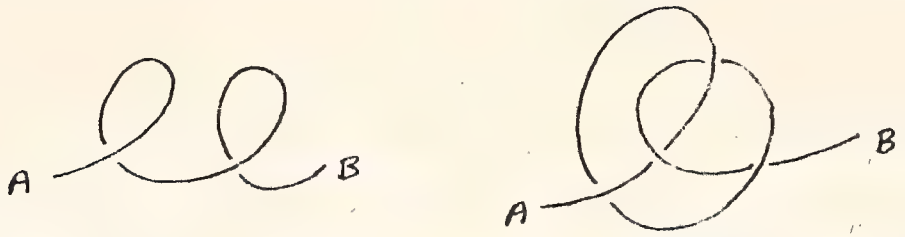
(b) the second half of a word is obtained by interchanging O and U in the first half of the word.

Any other observations?

Are statements (a) and (b) true of loop words in general?

Experiment with other loops - now!

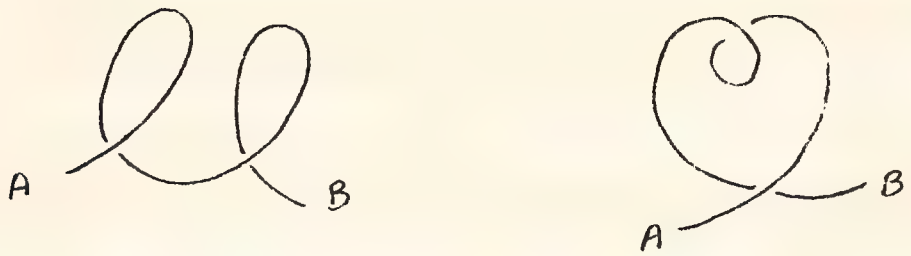
Statement (a) seems self-evident if we notice that each crossing must contribute an O and a U to the word making the number of letters in the word equal to twice the number of crossings in the loop. On the other hand, the counter-examples of Figure 3 show that (b) is not true of all words.



OUOU OUOUOUOU

Fig. 3

We have seen that the Os and Us of a loop word pair off and that each pair corresponds to a particular crossing. Figure 4 shows two different loops which are both described by OUOU : the pairing of O and U is shown in both cases.



OUOU OUOU

Fig. 4

Can you draw two loops corresponding to the word OOUU and having the pairings OOUU and OOUU respectively?

How about other ways of describing loops? We could, for example, number each crossing as shown in Figure 5.

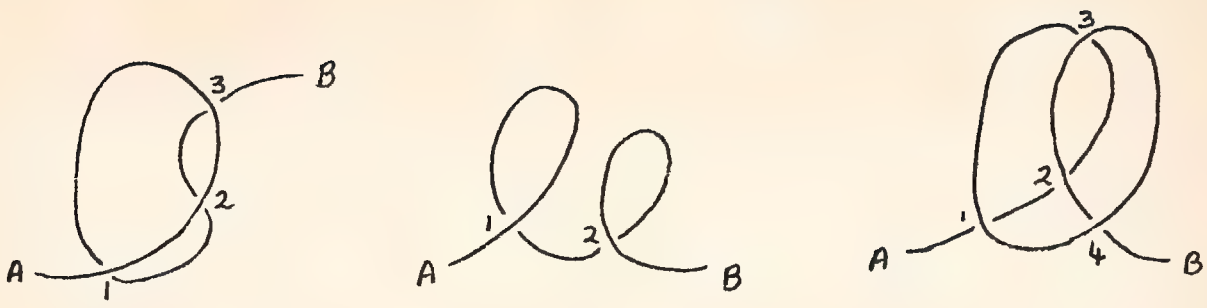


Fig. 5

The system adopted is that of allocating 1 to the first crossing encountered, starting from A , and then numbering in succession each <u>new</u> crossing after then. The numerical descriptions of the loops are:

Figure 5 (a) 123123
Figure 5 (b) 1122
Figure 5 (c) 12314324

At this stage the reader is left to investigate further the properties of loops and their descriptions.

Some questions:

What other ways are there of representing loops by strings of symbols?

Does the word and numerical description together provide a unique definition of a loop?

Is there a method of determining whether the loop is knotted by considering the word or numerical description or both?

Adapted from <u>Fifteen Starters for the Secondary Classroom</u>, a pamphlet published by the Association of Teachers of Mathematics.

FOR YOU TO LEARN

Your teachers stir you to remember the vital relationships

	θ	0^o	30^o	45^o	60^o	90^o
Sin	θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

But consider this table of relationships

	θ	0^o	30^o	45^o	60^o	90^o
Sin	θ	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$

- any help to you?

PROBLEMS

Solutions to Junior Problems will be accepted only from students in third or lower secondary years. Solutions of problems in the Open Section will be accepted from students in any grade from Matriculation downwards. Some, but not all problems in the Open Section require more advanced knowledge than those in the Junior Section.

All solutions sent in must be the result of the solvers' unaided efforts.

Where a solution is obtained by the co-operation of two participants, the solution should be jointly signed by both authors. Our published solutions sometimes have to be abbreviated, but solutions sent in should be given fully.

Problems in this issue (Vol.13, No.2). We hope to publish these solutions and the names of those who sent in good solutions in Vol.13, No.3. Solutions should be in the hands of the Editor within one month of receiving the Journal.

IMPORTANT. The solution of each problem must be written on a separate sheet, but the sheets may be sent in one envelope. Each sheet should be signed by the author(s), giving full name (in block letters), name of school and grade. You may send in solutions to one problem only or to any number of problems.

Solutions should be sent to The Editor, S.A. Mathematics Journal, Mathematics Department, University of Adelaide, Adelaide 5001.

(In some cases, Mathematical Associations outside South Australia have made alternative arrangements for the marking of solutions.)

JUNIOR PROBLEMS

8. Prove that

$$\frac{a^3+b^3}{2} > \left(\frac{a+b}{2}\right)^3$$

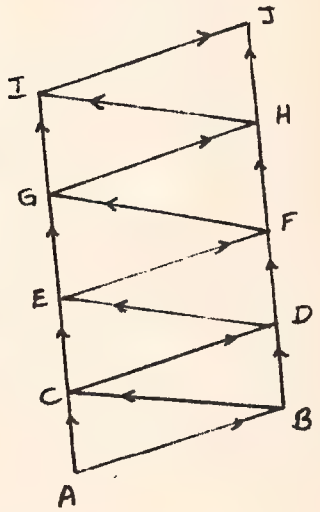
9. Prove that the sum of the 'legs' of a right-angled triangle does not exceed the length of the diagonal of the square on the hypotenuse.

10. AC and BD are diagonals of a regular pentagon ABCDE. Show that the length of the longer segment of each diagonal is equal to the length of a side of the pentagon.

11. Show that $n(n+1)(2n+1)$ is exactly divisible by 6.

12. Any three digit number abc is divisible by 7 if and only if $2a+3b+c$ is divisible by 7. Why is this so? Can you generalize this to a rule for four or more digits?

13. ABCDEFGHIJ is a zig-zag track up a mountainside. At each labelled point other than J a steep alternative track branches off along AI or BJ. How many routes lead from A to J?



14. A cardboard circle is cut along a diameter and a cone is formed out of one of the semicircular pieces. Now suppose that another cardboard circle is fitted to the base of the cone. This base circle is then cut along a diameter and a cone formed from one of the halves. What is the ratio of the volumes of the two cones?

OPEN SECTION

08. Show that by multiplying four consecutive numbers together and adding 1 we get a perfect square.

09. If p, q, p_1, q_1 are real numbers, and

$$pp_1 = 2(q+q_1),$$

show that at least one of the equations

$$x^2 + px + q = 0$$

$$x^2 + p_1x + q_1 = 0$$

has two real roots.

010. Find all integer solutions of $x+y = xy$.

011. Fourteen married couples live in a migrant hostel: five Italian couples, two Hungarian couples, three Yugoslav couples and four Greek couples. They decide to select a committee of seven from amongst themselves. In how many ways can this be done if each nationality is to be represented, and the committee is to have at least two women as members?

012. For what values of the common ratio can three successive terms of a geometric progression of positive numbers be sides of a triangle?

013. Prove that the numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ cannot be terms of a single arithmetic progression.

014. (i) The equation of the curve H is

$$y = \frac{1}{x}.$$

Determine the shortest distance from the origin O to the curve.

(ii) The equation of the curve L is

$$(x^2+y^2)^2 = 4xy$$

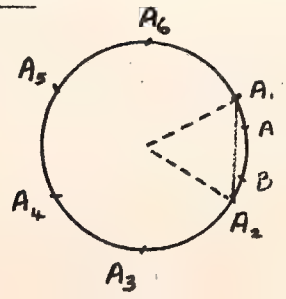
Let OPQ be a ray from the origin intersecting the curve L in P and the curve H in Q. Find a relationship between lengths OP, OQ, and hence sketch the curve L.

SOLUTIONS TO PROBLEMS IN VOL. 13, NO. 1

JUNIOR SECTION

1. Given a circle of radius 1 metre, and seven points lying on its circumference, show that there are two of these points which are less than 1 metre apart.

SOLUTION



Divide the circumference of the circle centre O radius 1 metre into 6 equal arcs by the points $A_1, A_2, A_3, A_4, A_5, A_6$.

Let S_1 = arc A_1A_2 ~ $\{A_2\}$

S_2 = arc A_2A_3 ~ $\{A_3\}$

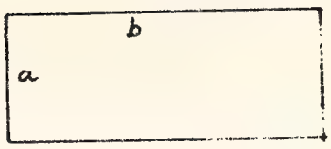
$\vdots$

S_6 = arc A_6A_1 ~ $\{A_1\}$

These sets are six disjoint sets which taken together make up the whole circumference of the circle. As there are seven given points lying on the circumference of the circle two must lie in one of the S_i's, say S_1. Let the two points be A and B. By construction $AB < A_1A_2$. And A_1A_2 = 1 metre = radius of circle. We have therefore found two points which are less than 1 metre apart.

2. Show that a rectangle of given perimeter has greatest area when its sides are equal.

SOLUTION



Let a rectangle have sides a and b units. Then it has perimeter $P = 2(a+b)$ and area $A=ab$.

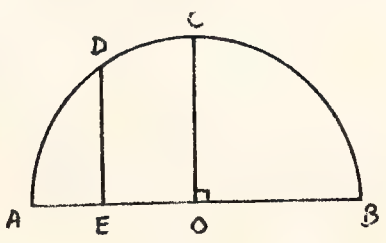
But $ab = \frac{1}{4}[(a+b)^2-(a-b)^2]$

Hence $A = \frac{1}{4}\left[\left[\frac{P}{2}\right]^2 - (a-b)^2\right]$.

If P is fixed, then A will be largest when $(a-b)^2 = 0$, that is when $a=b$. Thus a rectangle with a given perimeter P will have largest area if the rectangle is a square.

3. AB is the diameter of a semicircular arc and C is the midpoint of the arc. D is another point on the arc and E is its perpendicular projection on the diameter. Prove that $AC^2 = CE^2 + DE^2$.

SOLUTION



Let O be the centre of the circle of which AB is a diameter. In the diagram, using Pythagoras' theorem, we have

$AC^2 = AO^2+OC^2 = 2OC^2$, $(OC=OA)$ (1)

$DE^2 = DO^2-EO^2 = CO^2-EO^2$. $(CO=DO)$

$EC^2 = EO^2+CO^2$

Hence $DE^2+EC^2 = CO^2-EO^2+EO^2+CO^2 = 2CO^2$ (2)

And so by (1) and (2), $AC^2 = DE^2+EC^2$

4. Let a,b,c be positive numbers such that abc=1. Calculate the value of

$$S = \frac{a}{ab+a+1} + \frac{b}{bc+b+1} + \frac{c}{ca+c+1} .$$

SOLUTION

In S, since abc = 1, substitute $a = \frac{1}{bc}$, then

$$S = \frac{\frac{1}{bc}}{\frac{1}{bc}b + \frac{1}{bc} + 1} + \frac{b}{bc+b+1} + \frac{c}{c\,\frac{1}{bc} + c + 1}$$

$$= \frac{1}{b + 1 + bc} + \frac{b}{bc+b+1} + \frac{bc}{1 + bc + b}$$

$$= \frac{1 + b + bc}{1 + b + bc} = 1$$

5. Show that in any set of 10 consecutive integers there is at least one integer which is not divisible by 2,3,5 or 7.

SOLUTION

Among any 10 consecutive integers, there are 5 which are even (divisible by 2). Of the 5 remaining odd numbers, there are at most two divisible by 3, one divisible by 5. and at most one divisible by 7. This leaves at least one odd number in this list of 10 which is not divisible by either 2,3,5 or 7.

6. There is a house with the interesting property that if you stand in any of its rooms you can count an even number of doors.

Before the owner goes to bed each night, he locks all the windows, the front door, the back door and the side door. One morning he realized that his cutlery had been stolen; but when he checked the windows and doors, he saw that none of them had been forced open. How did the burglar get in?

SOLUTION

If one stands in a room of this peculiar house, one can count an even number of door knobs. This is because there is an even number of doors leading out of each room. Let the number of door knobs inside the house be N; then N must also be even, because it is the sum of a collection of even numbers. Let K be the number of door knobs which belong to doors

leading from one room into another. Then K is even, because there are two door knobs for each such door. Then N-K is even. But this is the number of door knobs of doors leading to the outside of the house. Hence the house must have an even number of doors on the outside. The owner claims that he always locked the front door, the back door and the side door, that is three doors. Therefore there must have been a fourth door which he did not lock.

7. Let p be the smallest prime divisor of an integer n, and suppose that $p > \sqrt[3]{n}$. Show that n/p is also prime.

SOLUTION

Suppose that $\frac{n}{p}$ is not a prime. Then it has at least two prime factors q,r ($q \leq r$ say), and these must also be factors of n.

Thus $pqr \leq n$ (1)

But p is the smallest prime divisor of n, so

$p \leq q \leq r$

Hence $p^3 \leq pqr \leq n$ by (1)

Since $p > \sqrt[3]{n}$, we have arrived at a contradiction.

OPEN SECTION

0.1. Let f be a bounded real valued function defined on $[0,1] = \{x \in R;\ 0 \leq x \leq 1\}$ with the property:

$$f\left(\frac{x+y}{2}\right) \leq \tfrac{1}{2}(f(x) + f(y))$$

for all $x,y \in [0,1]$. Suppose further that $f(0) = f(1) = 1$.

Show that $f(x) \leq 1$ for all $x \in [0,1]$.

SOLUTION

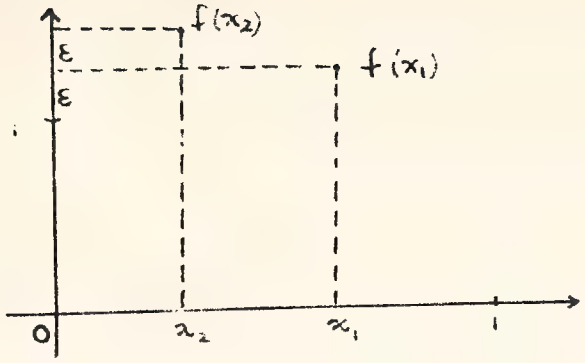
Suppose that there is a point x_1 with $0 < x_1 < 1$ such that

$$f(x_1) > 1$$

Let $f(x_1) = 1 + \varepsilon$ where $\varepsilon > 0$.

Then there is a point x_2, $0 < x_2 < 1$ such that x_1 is the midpoint of $[x_2,1]$ or $[0,x_2]$ and

$$f(x_2) \geq 1 + 2\varepsilon$$



If this did not hold, that is if $f(x_2) < 1+2\varepsilon$,

then $$f\left(\frac{x_2+1}{2}\right) \leq \tfrac{1}{2}f(x_2) + \tfrac{1}{2}f(1)$$

$$< \tfrac{1}{2}(1+2\varepsilon) + 1 = 1+\varepsilon = f(x_1)$$

But $\frac{x_2+1}{2} = x_1$, hence $f(x_1) < f(x_1)$ which is absurd. (We assumed here that $x_1 \geq \frac{1}{2}$). In a similar way we can find x_3 such that

$$f(x_3) \geq 1+2^2\varepsilon$$

and in general, we can find an x_n such that

$$f(x_n) \geq 1+2^{n-1}\varepsilon.$$

As $n\to\infty$, $f(x_n) \to \infty$ as $\varepsilon>0$, but this cannot happen since f is assumed to be bounded. Thus $f(x) \leq 1$ for all $0<x<1$.

02. Show that $\log_{10}2$ is an irrational number.

SOLUTION

Suppose that $\log_{10}2$ is rational. Let

$$\log_{10}2 = \frac{p}{q} \qquad (*)$$

where p and q are positive integers. Expressing (*) without logs yields

$$2 = 10^{\frac{p}{q}}$$

that is

$$2^q = 10^p$$

But as $p\geq 1$ it follows that since 5 divides 10^p, 5 also divides 2^q; but this is absurd. Hence (*) can never be true. Hence $\log_{10}2$ is irrational.

03. The real valued function $f(x)$ is defined for all x. Furthermore, there exists a fixed number a such that for all x

$$f(x+a) = \tfrac{1}{2} + \sqrt{f(x) - [f(x)]^2}$$

Show that $f(x)$ is periodic. (That is there exists a real number b such that for all x

$$f(x+b) = f(x).)$$

Construct an example of such a function.

SOLUTION

Firstly note that for all real x

$$f(x+a) = \tfrac{1}{2} + \sqrt{f(x) - [f(x)]^2} \geq \tfrac{1}{2}$$

hence $f(x) = f(x-a+a) \geq \frac{1}{2}$ for all real x.

Thus $f(x) - [f(x)]^2 = f(x)[1-f(x)] \geq 0$ and since $f(x) \geq \frac{1}{2} > 0$

$$1 - f(x) \geq 0 \quad \text{so} \quad f(x) \leq 1.$$

The range of $f(x)$ is then a subset of $[\frac{1}{2},1]$.

Form $f(x+2a) = f(x+a+a)$

$$= \tfrac{1}{2} + \sqrt{f(x+a) - [f(x+a)]^2}$$

$$= \tfrac{1}{2} + \sqrt{f(x+a)[1-f(x+a)]}$$

where

$$1 - f(x+a) = \tfrac{1}{2} - \sqrt{f(x) - [f(x)]^2}$$

hence

$$f(x+a)(1 - f(x+a)) = \tfrac{1}{4} - f(x) + [f(x)]^2$$

$$= [f(x) - \tfrac{1}{2}]^2$$

So

$$f(x+2a) = \tfrac{1}{2} + \sqrt{(f(x) - \tfrac{1}{2})^2}$$

$$= \tfrac{1}{2} + (f(x) - \tfrac{1}{2})$$

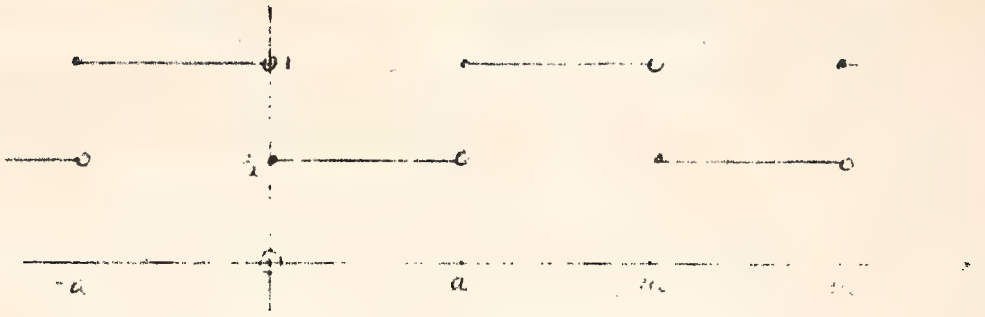
since $f(x) \geq \frac{1}{2}$.

Thus $f(x+2a) = f(x)$, that is $f(x)$ is periodic with period $2a$.

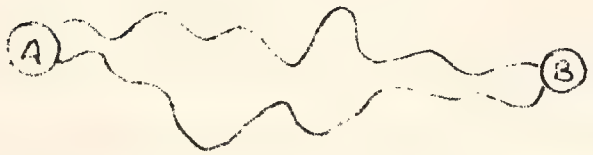
We may construct the following example (step-type function)

$$\text{Let } f(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x < a \\ 1 & \text{if } a \leq x < 2a \end{cases}$$

and extend $f(x)$ periodically outside $[0,2a]$.



04. In a certain country there are two towns A and B, separated at some distance. There are two very narrow non-intersecting paths leading from A to B.



One day two men set out on a walk from A to B, one on each road. They held between them throughout the journey a rope of length 2ℓ metres, and both managed to reach B without losing hold of the rope. Of course the rope may have been slack most of the time and it is not assumed that both men were necessarily travelling at the same speed. On the next day two large waggons travelled along these paths. The waggons were rather peculiar in shape. Looking from above they looked like a circle of radius ℓ metres. One waggon set out from A to reach B, the other set out from B to reach A. Show that to complete their journeys the waggons either collide or touch; in other words to complete their respective journeys one waggon will be forced off its path.

SOLUTION

Let us assign coordinates to each point on path 1 and each point on path 2. On path 1 we let A be represented by 0 and B represented by 1 and each point on path 1 by a number x between 0 and 1. On path 2 we also let A be represented by 0 and B by 1 and each point between them on path 2 by a number y also between 0 and 1. Consider the first day's events. At any given instant of time we can assign an ordered pair (x,y) being the positions of man 1 on path 1 and the position of man 2 on path 2. When both men are at A the situation is represented by (0,0); whereas when both men are at B, the situation is represented by (1,1). As time progresses and both men proceed from A to B their wanderings can be represented on a graph (fig 1.)

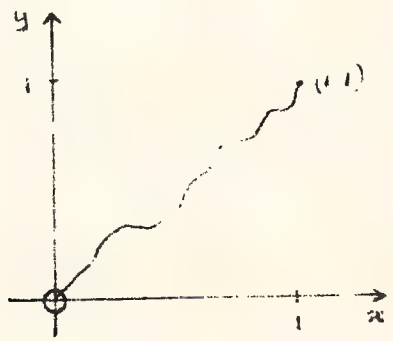


fig. 1

On the second day, we deal with waggons. Since one waggon started at A, the other at B, this situation is represented by (0,1). The final position may be that waggon 1 ends up at B, and waggon 2 ends up at A. This situation is represented by (1,0). The intervening journey can be represented on a graph (fig 2.)

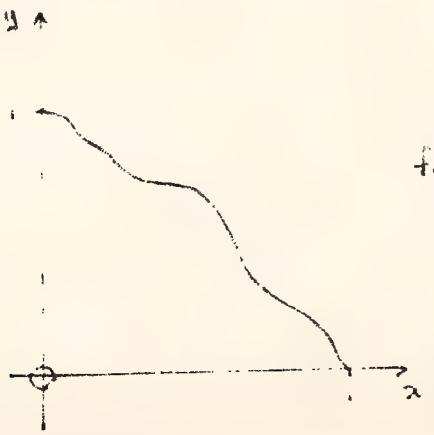
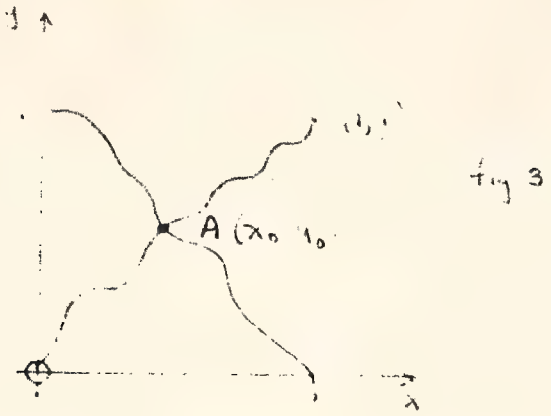


fig. 2

Let us draw these two graphs on the same axes.



Notice that the two graphs intersect at $A(x_0,y_0)$. What does this mean? It means that on day 1 man 1 was at x_0 when man 2 was at y_0 (on paths 1 and 2 respectively); which means in turn that the distance between x_0 and y_0 is less than or equal to 2ℓ. It means also that on the second day that waggon 1 was at x_0 when waggon 2 was at y_0 (on paths 1 and 2 respectively); and by what has just been said, this means the carts must either touch or collide.

05. Show that if a is a positive integer and p is a prime number then

$$a^p \equiv (a-1)^p + 1 \pmod{p}.$$

Hence show that $a^p \equiv a \pmod{p}$ for all positive integers a.

SOLUTION

Lemma: For all prime-numbers p, the binomial coefficient

$$\binom{p}{r}, \quad r \neq 0 \quad r \neq p$$

is a multiple of p, where

$$\binom{p}{r} = \frac{p(p-1)\ldots(p-r+1)}{1.2.3\ldots r}$$

Proof: The numerator of $\binom{p}{r}$ is divisible by the denominator since $\binom{p}{r}$ is an integer. But p does not occur amongst the factors of the denominator, hence the denominator must divide $(p-1)(p-2)\ldots(p-r+1)$. Thus

$$\binom{p}{r} = p.\ \frac{(p-1)(p-2)\ldots(p-r+1)}{1.2.3\ldots r} = mp$$

where m is an integer.

Now $a^p = (a-1+1)^p$

$$= (a-1)^p + \binom{p}{1}(a-1)^{p-1} + \binom{p}{2}(a-1)^{p-2} + \ldots$$

$$\ldots + \binom{p}{p-1}(a-1) + 1 \qquad (*)$$

where we have used the Binomial Theorem. Since by the above lemma all the terms on the right hand side of (*) except $(a-1)^p$ and 1 are divisible by p we may write

$$a^p = (a-1)^p + 1 + kp \quad \text{(where } k \text{ is an integer)}$$

that is $a^p \equiv (a-1)^p + 1 \pmod p$

We may now apply this result, substituting $a-1$ for a obtaining

$$(a-1)^p \equiv (a-2)^p + 1 \pmod p$$

hence

$$a^p \equiv (a-2)^p + 2 \pmod p$$

Continuing in this manner, we obtain finally,

$$a^p \equiv (a-a)^p + a \pmod p$$

That is $a^p \equiv a \pmod p$ for all positive integers

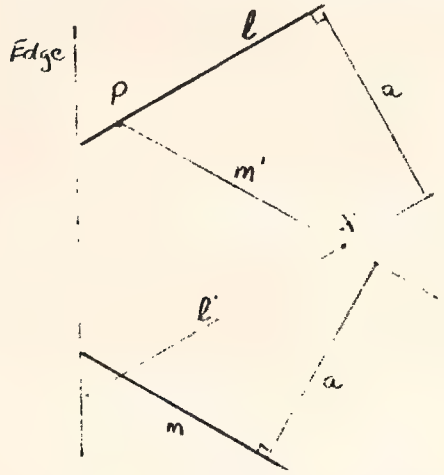
Note: This result is known as Fermat's (little) theorem.

06. Given two straight lines which are not parallel but do not meet on the sheet of paper on which they are drawn, make a construction on the paper for the bisector of the angle between them.

SOLUTION

Since the angular bisectors are the loci of the points equidistant from two given lines, it suffices to construct 2 points on the bisector. The construction may be carried out as follows: From one point P of the given line ℓ, construct a line m' parallel to m. Drop a perpendicular between these two parallels and assume that the length of this perpendicular is a. Now construct a line ℓ' para-

llel to ℓ and distance a from it. Let X be the point of intersection of the two parallel lines constructed.



Thus X is at a distance a from each of the lines ℓ and m, and hence must be on the internal bisector of the angle formed by ℓ and m. By a similar construction obtain a second point Y. Join X,Y to obtain the bisector.

(Note: the external bisector may be found similarly, by drawing m'' at a distance a on the other side of m, etc.)

07. ABCDE is a convex pentagon. In how many ways is it possible to colour the edges and diagonals red or blue if no triangle determined by any three of the vertices is to have all its sides the same colour?

SOLUTION

Since the perimeter of each triangle determed by three vertices of the pentagon consists of either two sides and a diagonal, or two diagonals and one side, a solution is certainly obtained by colouring <u>all</u> sides by one colour and <u>all</u> the diagonals by the other colour. (Fig.1. The heavy lines represent one colour, e.g. the red, the dotted lines the other colour.)

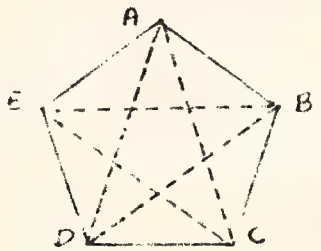


Fig. 1

Next seek a solution where the colouring of the perimeter is mixed. There will certainly be <u>at least</u> one pair of adjacent sides coloured the same, since the number of the sides is odd. Assume now that it is possible to solve the problem by having two pairs of adjacent sides coloured red and blue respectively, as shown on fig. 2: sides AB and BC red, CD and DE blue. The possibility or impossibility of the solution is not affected by the decision of colouring EA red or blue: in either

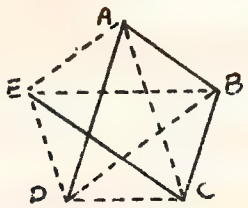


Fig. 2

case we have basically the same situation: 2 adjacent sides of one colour, and the other three of the other colour. Whichever choice is made however, the subsequent steps in colouring the diagonal cannot be chosen freely any more. Proceed in the following order: AC (blue, refer to triangle ABC),

EC (red, triangle EDC),

AD (red, triangle AED),

BE (blue, triangle BCE),

BD (blue, triangle BAD).

We thus arrive at triangle BED, all blue, hence no solution is obtained this way.

Thus if any solution is found by a mixed colouring of the perimeter, we can only have one pair of adjacent sides coloured the same way. The situation is illustrated in Fig. 3.

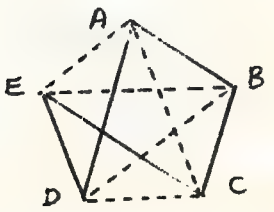


Fig. 3

Finding the colouring of the diagonals now in the order:
AC (blue, triangle ABC),
CE (red, triangle CAE),
BE (blue, triangle BCE),
AD (red, triangle ACD),
DB (blue, triangle DAB).

It is easy to check the remaining triangles to see that this is indeed a solution.

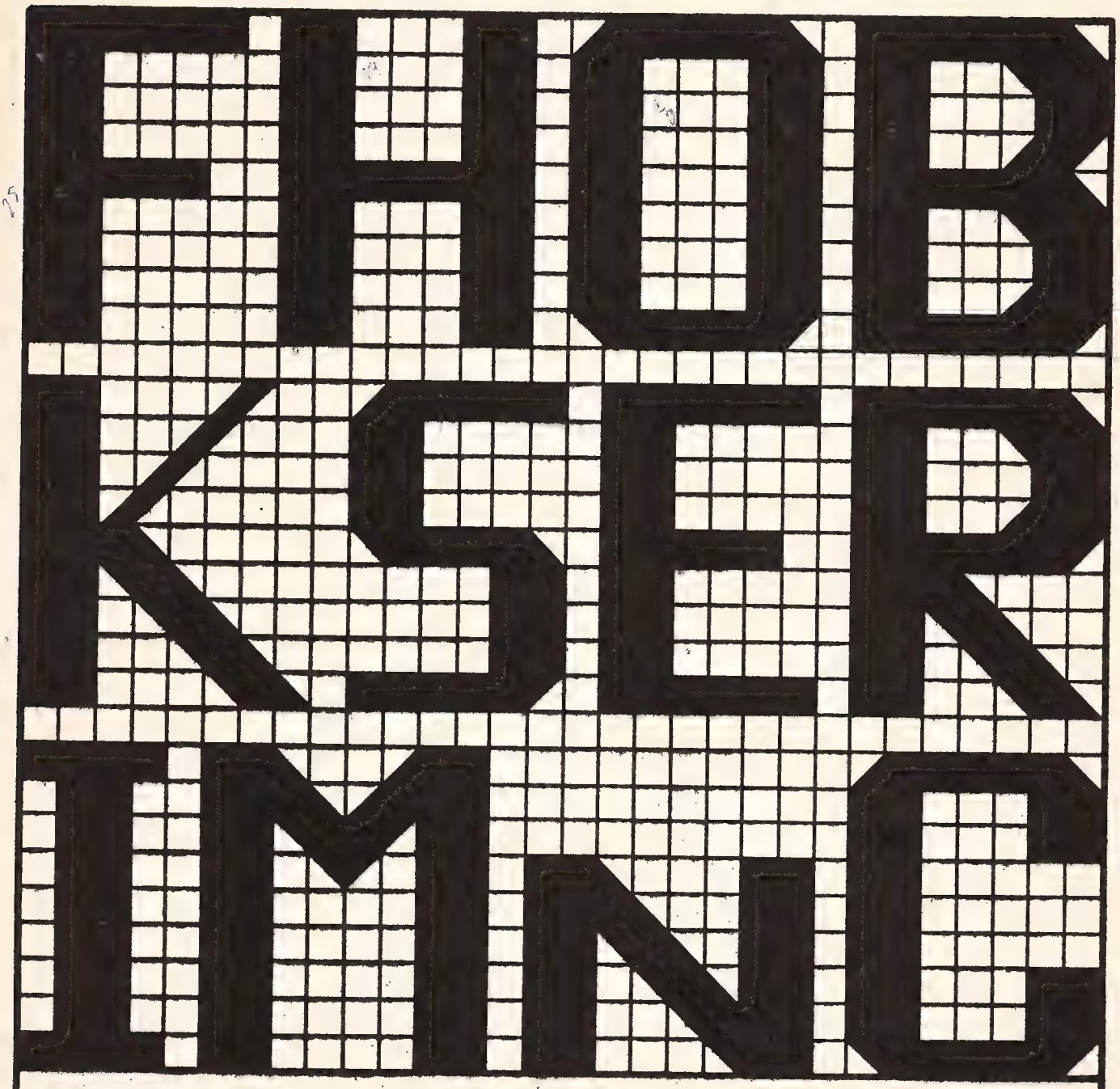
Hence there are only two basic types of solutions, shown in fig.1 and fig.3 respectively. Now in fig.3 the two adjacent red sides may be chosen in 5 different ways. Changing red for blue will give a new colouring in each case. Thus the total number of solutions is

$$2 \times (5+1) = 12.$$

A SQUARE DEAL

DIRECTIONS: *Figure out the AREA of any letter below and find your answer in the code at the bottom of the page. Each time the answer appears in the code, write the letter above it.*

KEEP WORKING UNTIL YOU HAVE DECODED THE MESSAGE, WHICH IS A "GOOD YOLK"!



TITLE
A GOOD
YOLK

34–39–22–34–31½–29–32–28–34–40–44–29–25–41½–40–44

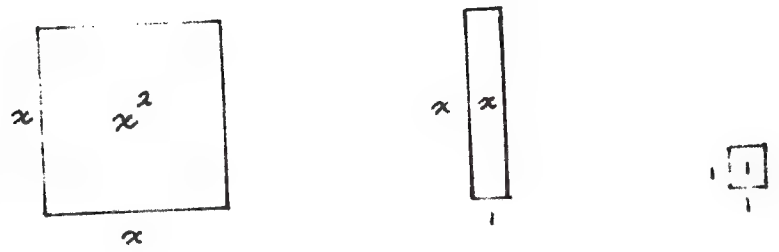
45–41½–40–31½–29–32–39–40–44–29–28

REPRINTED FROM MATHIMAGINATION

TRYING TRINOMIALS

All too often when we study algebra we can forget that our variables represent numbers and that the expressions we are manipulating may correspond to a physical object. Consider the following discussion which related trinomials and the factorization of trinomials to a physical model.

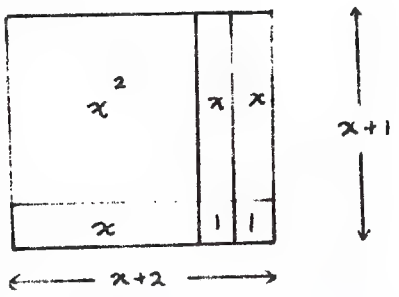
Draw on paper and cut out several of each of the following.



Select x as any convenient whole number (say 5, 6, 7, ..., 10) and let the length of the side of the larger side be x cm.

Consider the problem of factorizing $x^2 + 3x + 2$.

From your supply of paper pieces choose one large square ($1.x^2$), three rectangles ($3.x$) and 2 small squares (2.1). Can you arrange them to form a rectangle? What are the dimensions of the rectangle?



The area of the rectangle formed is $x^2 + 3x + 2$. The dimensions are $x + 2$ by $x + 1$.

This implies $x^2 + 3x + 2 = (x + 2)(x + 1)$.

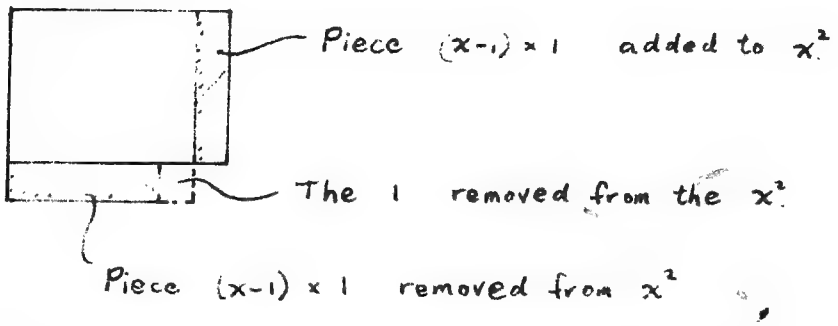
Can you find an alternative manner in which to arrange these six pieces to form a rectangle?

The fact that you can't indicates that the physical model consisting of the sheets of paper can be used to not only demonstrate factorization of trinomials but also the uniqueness of this factorization.

Try a few for yourself. Factorize $2x^2 + 3x + 1$, $2x^2 + 5x + 2$, $3x^2 + 4x + 1$. Can you see why $x^2 + 4x + 4$ is termed a perfect square?

When the coefficients in the trinomial are negative we must be prepared to modify our model a little. Often to create a rectangle from the given pieces, rectangles must be cut from the large squares and alternatively, small squares cut from the rectangles.

Can you see that the following diagram illustrates that $x^2 - 1 = (x - 1)(x + 1)$?



Can you see how to illustrate the factorization

$$x^2 + x - 2 = (x + 2)(x - 1)$$

Now you understand your model, try to illustrate some further factorization of trinomials.

THE TEN MATHEMATICAL EQUATIONS THAT CHANGED THE FACE OF THE EARTH

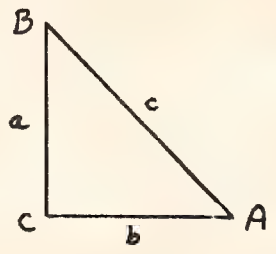
This eye-catching title seems to be a good title to head an article on mathematics, but believe it or not, it is the title of a series of stamps issued by Nicaragua in 1971.

Each of the 10 stamps bears an equation and a picture related to an application of the equation. Unusual? Even more unusual is the fact that each stamp has explanatory paragraphs written on their reverse side!

The first stamp features the equation $1 + 1 = 2$. It is the most appropriate equation to represent the achievement of primitive man in learning to count. Man began to count using simple objects such as small stones to represent the items he was counting. He used a method of one-to-one correspondence, not easy to represent by an equation.

x x x x x x Morning Evening x x x x x
o o o o o o o o o o o

One sheep lost!



The famous equation of Pythagoras, $a^2 + b^2 = c^2$ which relates the lengths of the sides of a right triangle is featured on the second stamp. Although the equation is associated with the name of Pythagoras (570-497 BC), particular cases of the theorem were known to the Egyptians some one thousand years earlier. Pythagoras and his followers are thought to be the first to prove that the relationship is true for all right triangles.

Logarithms are featured on the third stamp through Napier's relationship for logarithms

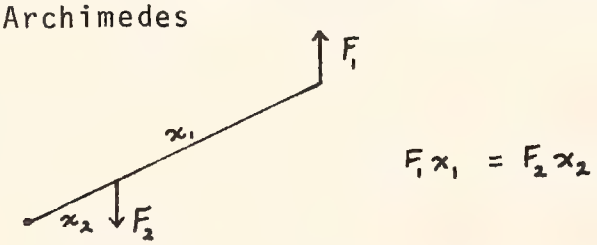
$$e^{\ln N} = N \quad \text{or} \quad \ln e^{N} = N$$

The invention of logarithms by the Scotsman John Napier during the seventeenth century had an impact on mathematics and science as important as the impact of the computer on current day studies. Logarithms allowed mathematicians and scientists to multiply and divide much faster than they had been previously able. As a result branches of science depending on lengthy calculations, such as astronomy and navigation, were able to develop more easily.

The next two stamps in the series feature equations from applied mathematics. One is Newton's Law of Gravitation

$$f = \frac{Gm_1 m_2}{r^2}$$

from which the force of attraction between two masses can be calculated. With this equation Newton was able to demonstrate that all masses attract one another, the force of attraction depending on the size of the masses and their distance apart. The other is due to Archimedes



and shows the relationship between two forces and their distances from a fulcrum or point of balance. This principle is used far more than we realize, wheel barrows, weighing scales and door handles being a few examples.

The other five stamps feature equations not so well known. These are Einstein's mass energy equation ($E = mc^2$), Boltzmann's entropy equation, Tsiolkovskii's equation for rocket speeds, de Broglie's wavelength equation and Maxwell's equation for electromagnetic radiation.

ON SOLUTIONS RECEIVED TO PROBLEMS IN TRIGON VOLUME 13, NUMBER 1

Junior Problems

There was a disappointing response to this section, except by H. J. Seppelt (SPSC, third year) who provided good solutions to Problems 2 and 7, and also solved Problems 3, 4, 5 and 6. We hope you have more competition next time, H. J.!

Open Section

01 Incomplete solutions were received from Chia Mon Ming, Matthew Webber and Mark Yudkin. The arguments used here were intuitively plausible but unjustified.

02 We particularly liked the solution by D. A. C. Robertson (P.A.C., fourth year). Good solutions were also received from Chia Mon Ming, Matthew Webber and Mark Yudkin.

03 Partial solutions were received from Matthew Webber and Mark Yudkin.

04 Solved by Matthew Webber and Mark Yudkin.

05 A good solution was received from D. A. C. Robertson. Partial solutions were sent in by Chia Mon Ming, Matthew Webber and Mark Yudkin.

06 Our choice here is the solution sent in by Chia Mon Ming (Matriculation, Blackfriars Priory School). Good solutions by various methods were also received from D. A. C. Robertson, Matthew Webber and Mark Yudkin.

07 Good solutions were sent in by D. A. C. Robertson and Matthew Webber. Mark Yudkin sent in a partial solution.

GEORGE POLYA

Born in Budapest, 13 December 1887. Now Professor Emeritus of Mathematics at Stanford University, California. An outstanding research mathematician and an inspiring teacher, he is well known for his book "How to Solve It".